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Emmanuel Humbert, Yannick Privat, Emmanuel Trélat

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Emmanuel Humbert* Yannick Privat† Emmanuel Trélat‡

Abstract

We consider the wave equation on a closed Riemannian manifold. We observe the restriction of the solutions to a measurable subset ω along a time interval $[0,T]$ with $T>0$. It is well known that, if ω is open and if the pair (ω,T) satisfies the Geometric Control Condition then an observability inequality is satisfied, comparing the total energy of solutions to their energy localized in $\omega\times(0,T)$. The observability constant $C_T(\omega)$ is then defined as the infimum over the set of all nontrivial solutions of the wave equation of the ratio of localized energy of solutions over their total energy.

In this paper, we provide estimates of the observability constant based on a low/high frequency splitting procedure allowing us to derive general geometric conditions guaranteeing that the wave equation is observable on a measurable subset ω. We also establish that, as $T\to+\infty$, the ratio $C_T(\omega)/T$ converges to the minimum of two quantities: the first one is of a spectral nature and involves the Laplacian eigenfunctions; the second one is of a geometric nature and involves the average time spent in ω by Riemannian geodesics.

Keywords: wave equation, observability inequality, geometric control condition.

Contents

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1 Introduction

Let (Ω, g) be a compact connected Riemannian manifold of dimension n without boundary. The canonical Riemannian volume on Ω is denoted by v_g, inducing the canonical measure dv_g. Measurable sets are considered with respect to the measure dv_g.

Consider the wave equation

$$\partial_{tt} y - \triangle_g y = 0 \qquad \text{in } (0,T) \times \Omega \tag{1}$$

where $\triangle_g$ stands for the usual Laplace-Beltrami operator on Ω for the metric g. Recall that the Sobolev space $H^1(\Omega)$ as the completion of the vector space of C^∞ functions having a bounded gradient (for the Riemannian metric) in $L^2(\Omega)$ for the norm given by $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$ and that $H^{-1}(\Omega)$ is the dual space of $H^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$.

For every set of initial data $(y(0,\cdot), \partial_t y(0,\cdot)) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique solution $y \in \mathcal{C}^0(0,T; L^2(\Omega)) \cap \mathcal{C}^1(0,T; H^{-1}(\Omega))$ of (1).

Let $T > 0$ and let ω be an arbitrary measurable subset of Ω of positive measure. The notation χ_ω stands for the characteristic function of ω, in other words the function equal to 1 on ω and 0 elsewhere. The **observability constant** in time T associated to (1) is defined by

$$C_T(\omega) = \inf \left\{ J_T^\omega(y^0, y^1) \ \mid \ (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0,0)\} \right\} \tag{2}$$

where

$$J_T^\omega(y^0, y^1) = \frac{\int_0^T \int_\omega |y(t,x)|^2 \, dv_g \, dt}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2}. \tag{3}$$

In other words, $C_T(\omega)$ is the largest possible nonnegative constant C such that

$$C\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leqslant \int_0^T \int_\omega |y(t,x)|^2 \, dv_g(x) \, dt$$

for all $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such that $(y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1)$. The equation (1) is said to be *observable* on ω in time T if $C_T(\omega) > 0$. Note that, by conservation of energy, we always have $0 \leqslant C_T(\omega) \leqslant T$. It is well known that if ω is an open set then observability holds when the pair (ω, T) satisfies the *Geometric Control Condition* in Ω (see [1, 2, 28]), according to which every ray of geometric optics that propagates in Ω intersects ω within time T. This classical result will be slightly generalized to more general subsets ω within this paper. Let us also mention the recent article [14] where the authors provide sharp estimates of the observability constant at the minimal time at which unique continuation holds for the wave equation.

This article is devoted to establishing various properties of the observability constant. Our main results are stated in Section 2. We first show that, under appropriate assumptions on the observation domain ω, the limit of $C_T(\omega)/T$ as $T \to +\infty$ exists, is finite and is written as the minimum of two quantities: the first one is a *spectral quantity* involving the eigenfunctions of $-\triangle_g$ and the second one is a *geometric quantity* involving the geodesics of Ω. We then provide a characterization of observability (Corollary 1) based on a low/high frequency splitting procedure (Theorem 1) showing how observability can be characterized in terms of high-frequency eigenmodes. In turn, our approach gives a new proof of results of [1, 28] on observability. Finally, we investigate the case where there is a spectral gap assumption on the spectrum of $-\triangle_g$.

2 Statement of the results

Let $T > 0$ and let ω be a measurable subset of Ω.

Let $(\phi_j)_{j\in\mathbb{N}^*}$ be an arbitrary Hilbert basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\triangle_g$, associated with the real eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}^*}$ such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \to +\infty$. For every $N \in \mathbb{N}$, we define

$$C_T^{>N}(\omega) = \inf\{J_T^\omega(y^0, y^1) \mid \langle y^i, \phi_j\rangle_{(H^i)',H^i} = 0, \quad \forall i = 0,1, \quad \forall j = 1,\dots,N \\ (y^0,y^1) \in L^2(\Omega)\times H^{-1}(\Omega)\setminus\{(0,0)\}\} \quad (4)$$

with the convention that $H^0 = L^2$. Noting that $C_T(\omega) \leqslant C_T^{>N}(\omega) \leqslant C_T^{>N+1}(\omega)$ for every $N \in \mathbb{N}$, we define the "high-frequency" observability constant as follows.

Definition 1 (high-frequency observability constant)**.** *The* high-frequency observability constant *$\alpha^T(\omega)$ is defined by*

$$\alpha^T(\omega) = \lim_{N\to+\infty} \frac{1}{T} C_T^{>N}(\omega).$$

This limit exists since the mapping $\mathbb{N} \ni N \mapsto C_T^{>N}(\omega)$ is nondecreasing and is bounded[1].

Definition 2 (Spectral quantity $g_1(\omega)$)**.** *The* ***spectral quantity*** *$g_1(\omega)$ is defined by*

$$g_1(\omega) = \inf_{\phi\in\mathcal{E}} \frac{\int_\omega |\phi(x)|^2\, dv_g}{\int_\Omega |\phi(x)|^2\, dv_g},$$

where the infimum runs over the set $\mathcal{E}$ of all nonconstant eigenfunctions ϕ of $-\triangle_g$.

Main results on the observability constant $C_T(\omega)$

Theorem 1. *Given any $T > 0$ and any measurable subset $\omega \subset \Omega$, we have*

$$\frac{C_T(\omega)}{T} \leqslant \min\left(\frac{1}{2} g_1(\omega), \alpha^T(\omega)\right).$$

Moreover, if $\frac{C_T(\omega)}{T} < \alpha^T(\omega)$ then the infimum in the definition of C_T is reached: there exists $(y^0, y^1) \in L^2(\Omega)\times H^{-1}(\Omega)\setminus\{(0,0)\}$ such that

$$\frac{C_T(\omega)}{T} = J_T^\omega(y^0, y^1) > 0.$$

In what follows we are going to provide explicit estimates of $\alpha^T(\omega)$, thus yielding observability properties.

[1]This follows by conservation of the energy $[0,T] \ni t \mapsto \|\partial_t y(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla y(t,\cdot)\|^2_{L^2(\Omega)}$ for any solution y of (1).

Corollary 1. *We have $C_T(\omega) > 0$ if and only if $\alpha^T(\omega) > 0$.*

Note that this result is valid for any Lebesgue measurable subset ω of Ω and for any $T > 0$. Corollary 1 says that observability is a high-frequencies property, which was already known when inspecting the proofs of GCC in [1, 15], but the above equivalence with the notion of high-frequency observability constant, was never stated like that, up to our knowledge. Besides, our objective is also to investigate what happens for measurable subsets ω that are not open.

Remark 1. The results established in [1] are valid for manifolds having a nonempty boundary. Corollary 1 above is still true in this context but extending the results hereafter to such geometries would require a deeper study of $\alpha^T(\omega)$ on manifolds with boundary, which are beyond the scope of this paper

As a consequence of our techniques of proof, which are based on a concentration-compactness argument, we get the following large-time asymptotics of the observability constant $C_T(\omega)$.

Theorem 2 (Large-time observability)**.** *Given any $T > 0$ and any measurable subset $\omega \subset \Omega$, the limit*

$$\alpha^\infty(\omega) = \lim_{T\to+\infty} \alpha^T(\omega)$$

exists and we have

$$\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \min\left(\frac{1}{2}g_1(\omega), \alpha^\infty(\omega)\right). \tag{5}$$

Moreover, if $\frac{1}{2}g_1(\omega) < \alpha^\infty(\omega)$ then $g_1(\omega)$ is reached, i.e., the infimum in the definition of $g_1(\omega)$ is in fact a minimum.

Consequences of this result are given hereafter.

Characterization of the quantities $\alpha^T(\omega)$ and $\alpha^\infty(\omega)$

In what follows, we say that γ is a *ray* if γ is a Riemannian geodesic traveling at speed one in Ω. We denote by Γ the set of all rays of Ω.

Definition 3 (Geometric quantity $g_2(\omega)$)**.** *We define*

$$g_2^T(\omega) = \inf_{\gamma\in\Gamma} \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt \tag{6}$$

and

$$g_2(\omega) = \lim_{T\to+\infty} g_2^T(\omega). \tag{7}$$

The quantity $g_2^T(\omega)$ stands for the minimal average time spent by a geodesic γ in ω. Note that the mapping $T \mapsto g_2^T(\omega)$ is nonnegative, is bounded above by 1 and is subadditive. Hence the limit in the definition of $g_2(\omega)$ is well defined.

In [9], it has been shown how to compute the geometric quantity $g_2(\omega)$ have been established in the case where Ω is a square, $\triangle_g$ the Dirichlet-Laplacian operator on Ω and $\omega \subset \Omega$ is a finite union of squares.

Theorem 3 (Computation of $\alpha^T(\omega)$)**.** *Given any $T > 0$ and any measurable subset $\omega \subset \Omega$, we have*

$$\frac{1}{2}g_2^T(\mathring{\omega}) \leqslant \alpha^T(\mathring{\omega}) \leqslant \alpha^T(\omega) \leqslant \alpha^T(\overline{\omega}) \leqslant \frac{1}{2}g_2^T(\overline{\omega}).$$

Let γ be the support of a closed geodesic of Ω and set $\omega = \Omega \setminus \gamma$ (open set). Then $\alpha^T(\omega) = 1$ and $g_2^T(\omega) = 0$. Hence, the estimate given by Theorem 3 is not sharp.

Note however that, if ω is Jordan mesurable, i.e., if the Lebesgue measure of $\partial\omega = \overline{\omega} \setminus \mathring{\omega}$ is zero, then it follows from the definition of $C_T^{>N}$ that $C_T^{>N}(\omega) = C_T^{>N}(\overline{\omega})$ for every $N \in \mathbb{N}$. As a consequence, Theorem 3 can be improved in that case by noting that $\frac{1}{2} g_2^T(\mathring{\overline{\omega}}) \leqslant \alpha^T(\omega)$, under additional regularity assumptions on ω.

Corollary 2. *If the measurable subset ω satisfies the regularity assumption*

$$(\mathbf{H}) \qquad\qquad g_2^T(\Omega \setminus (\overline{\omega} \setminus \mathring{\omega})) = 1$$

then

$$2\alpha^T(\omega) = g_2^T(\mathring{\omega}) = g_2^T(\overline{\omega}) = g_2^T(\omega).$$

Many measurable sets ω satisfy Assumption $(\mathbf{H})$. Geometrically speaking, $(\mathbf{H})$ stipulates that ω has no *ray grazing*[2] $\partial\omega$ and sticking along it over a set of times of positive measure.
As a consequence of Corollary 1, Corollary 2 and Theorem 3, one has the following simple characterization of observability.

Corollary 3. *Let $T > 0$ and let $\omega \subset \Omega$ be a Lebesgue measurable subset of Ω.*

(i) If $g_2^T(\mathring{\omega}) > 0$ then $C_T(\omega) > 0$.

(ii) If $C_T(\overline{\omega}) > 0$ then $g_2^T(\overline{\omega}) > 0$.

(iii) Assume that ω satisfies the regularity assumption $(\mathbf{H})$. Then

$$g_2^T(\omega) > 0 \;\Leftrightarrow\; C_T(\omega) > 0.$$

The first item above is already well known (see [1, 28]): it says that, for ω open, GCC implies observability. Indeed, the condition $g_2^T(\mathring{\omega}) > 0$ is exactly GCC for $(\mathring{\omega}, T)$. As already mentioned, the article [1] also deals with manifolds with boundary, which is not the case in this article. Recovering the boundary case by the method we present here would require a deeper study of the quantity $\alpha^T(\omega)$ that we do not perform here. We also mention [2], where the authors prove that GCC is necessary and sufficient when replacing the characteristic function of ω by a continuous density function a in all quantities introduced above.

When there exist grazing rays sticking along $\partial\omega$ over a set of times of positive measure, the situation is more intricate. For instance, take $\Omega = \mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$, and take ω the open Northern hemisphere. Then, the equator is a trapped ray (i.e., it never meets ω) and is grazing ω. Therefore we have $g_2^T(\omega) = 0$ for every $T > 0$, while $C_T(\omega) = g_1(\omega) = g_1(\overline{\omega}) = g_2^T(\overline{\omega}) = 1/2$ for every $T \geqslant \pi$ (this follows immediately from computations done in [17]).

Note also that $g_1(\omega) > 0$ is not sufficient to guarantee that (1) is observable on ω. For instance, take $\Omega = \mathbb{T}^2$, the 2D torus, in which we choose ω as being the union of four triangles, each of them being at an corner of the square and whose side length is $1/2$. By construction, there are two trapped rays along $x = 1/2$ and $y = 1/2$ touching ω without crossing it over a positive duration. It follows that $g_2^T(\omega) = g_2(\omega) = C_T(\omega) = 0$ for every $T > 0$. Moreover, simple computations show that $g_1(\omega) > 0$.

From Theorem 2 and Corollary 2, one gets the following asymptotic result.

[2] Recall that a ray $\gamma \in \Gamma$ is grazing $\partial\omega$ at time t if $\gamma(t)$ is tangent to $\partial\omega$.

Corollary 4. *If the measurable subset ω satisfies* $(\mathbf{H})$ *then*

$$\lim_{T\to+\infty}\frac{C_T(\omega)}{T}=\frac{1}{2}\min\left(g_1(\omega),g_2(\overline{\omega})\right).$$

Remark 2. The above result echoes a result by G. Lebeau that we recall hereafter. In [18], the author considers the damped wave equation

$$\partial_{tt}y(t,x)-\triangle_g y(t,x)+2a(x)\partial_t y(t,x)=0 \tag{8}$$

on a compact Riemannian manifold Ω with a $\mathcal{C}^\infty$ boundary, where the function $a(\cdot)$ is a smooth nonnegative function on Ω. Given any $(y^0,y^1)\in H^1_0(\Omega)\times L^2(\Omega)$, for any $t\in\mathbb{R}$ we define

$$E_{(y^0,y^1)}(t)=\int_\Omega\left(|\nabla y(t,x)|^2+(\partial_t y(t,x))^2\right)dv_g$$

the energy at time t of the unique solution y of (8) such that $(y(0,\cdot),\partial_t y(0,\cdot))=(y^0,y^1)$. Let ω be any open set such that $a\geqslant\chi_\omega$ almost everywhere in Ω. If (ω,T) satisfies GCC then there exist $\tau>0$ and $C>0$ such that

$$E_{(y^0,y^1)}(t)\leqslant Ce^{-2\tau t}E_{(y^0,y^1)}(0) \tag{9}$$

for all $(y^0,y^1)\in H^1_0(\Omega)\times L^2(\Omega)$ (see [1, 7, 18]) and it is established in [18, Theorem 2] that the smallest decay rate $\tau(a)$ such that (9) is satisfied is

$$\tau(a)=\min\left(-\mu(\mathcal{A}_a),g_2(a)\right)$$

where $g_2(a)$ is the geometric quantity defined by (7) with χ_ω replaced by a, and $\mu(\mathcal{A}_a)$ is the spectral abscissa of the damped wave operator $\mathcal{A}_a=\begin{pmatrix}0 & \mathrm{Id}\\ \triangle_g & -2a(\cdot)\,\mathrm{Id}\end{pmatrix}$.

Remark 3 (Probabilistic interpretation of the spectral quantity $g_1(\omega)$)**.** The quantity $g_1(\omega)$ can be interpreted as an averaged version of the observability constant $C_T(\omega)$, where the infimum in (2) is now taken over random initial data. More precisely, let $(\beta^\nu_{1,j})_{j\in\mathbb{N}^*}$ and $(\beta^\nu_{2,j})_{j\in\mathbb{N}^*}$ be two sequences of Bernoulli random variables on a probability space $(\mathcal{X},\mathcal{A},\mathbb{P})$ such that

- for $m=1,2$, $\beta^\nu_{m,j}=\beta^\nu_{m,k}$ whenever $\lambda_j=\lambda_k$,
- all random variables $\beta^\nu_{m,j}$ and $\beta^\nu_{m',k}$, with $(m,m')\in\{1,2\}^2$, j and k such that $\lambda_j\neq\lambda_k$, are independent,
- there holds $\mathbb{P}(\beta^\nu_{1,j}=\pm1)=\mathbb{P}(\beta^\nu_{2,j}=\pm1)=\frac{1}{2}$ and $\mathbb{E}(\beta^\nu_{1,j}\beta^\nu_{2,k})=0$, for every j and k in $\mathbb{N}^*$ and every $\nu\in\mathcal{X}$.

Using the notation $\mathbb{E}$ for the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$, we claim that $\frac{T}{2}g_1(\omega)$ is the largest nonnegative constant C for which

$$C\|(y^0,y^1)\|^2_{L^2\times H^{-1}}\leqslant\mathbb{E}\left(\int_0^T\int_\Omega\chi_\omega(x)|y^\nu(t,x)|^2\,dv_g\,dt\right)$$

for all $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, where y^ν is defined by

$$y^\nu(t,x)=\sum_{j=1}^{+\infty}\left(\beta^\nu_{1,j}a_je^{i\lambda_jt}+\beta^\nu_{2,j}b_je^{-i\lambda_jt}\right)\phi_j(x),$$

where the coefficients a_j and b_j are defined by

$$a_j = \frac{1}{2}\left(\int_\Omega y^0(x)\phi_j(x)\,dv_g - \frac{i}{\lambda_j}\int_\Omega y^1(x)\phi_j(x)\,dv_g\right),$$
$$b_j = \frac{1}{2}\left(\int_\Omega y^0(x)\phi_j(x)\,dv_g + \frac{i}{\lambda_j}\int_\Omega y^1(x)\phi_j(x)\,dv_g\right)$$

for every $j \in \mathbb{N}^*$. In other words, y^ν is the solution of the wave equation (1) associated with the random initial data $y_0^\nu(\cdot)$ and $y_1^\nu(\cdot)$ determined by their Fourier coefficients $a_j^\nu = \beta_{1,j}^\nu a_j$ and $b_j^\nu = \beta_{2,j}^\nu b_j$. This largest constant is called *randomized observability constant* and has been defined in [26, Section 2.3] and [25, Section 2.1]. We also refer to [27] for another deterministic interpretation of $\frac{T}{2}g_1(\omega)$.

Remark 4 (Extension of Corollary 4 to manifolds with boundary.)**.** One could expect that a similar asymptotic to the one stated in Corollary 4 holds for the Laplace-Beltrami operator on a manifold Ω such that $\partial\Omega \neq \emptyset$, with homogeneous Dirichlet boundary conditions. For instance, in the 1D case $\Omega = (0,\pi)$, it is prove in [24, Lemma 1] by means of Fourier analysis that for every measurable set ω

$$\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dv_g = g_1(\omega) \qquad \text{with} \quad \phi_j(x) = \frac{1}{\sqrt{\pi}}\sin(jx).$$

In higher dimension, the problem is more difficult because we are not able to compute explicitly $\alpha^T(\omega)$ (see the proof of Theorem 3 where we use the Egorov theorem).

Spectral gap and consequences

Theorem 4. *Assume that the spectrum $(\lambda_j)_{j\in\mathbb{N}^*}$ satisfies the uniform gap property*

(UG) There exists $\gamma > 0$ such that if $\lambda_j \neq \lambda_k$ then $|\lambda_j - \lambda_k| \geqslant \gamma$.

Then for every measurable subset ω of Ω we have

$$\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \frac{1}{2}g_1(\omega).$$

As a consequence, thanks with Theorems 2 and 3, under (UG) we have

$$g_1(\omega) \leqslant g_2(\overline{\omega}) \tag{10}$$

for every measurable subset ω of Ω. Note that, without spectral gap, such an inequality obviously does not hold true in general: take Ω the flat torus and ω a rectangle in the interior of Ω (see [26, 25] for various examples).

Remark 5. Note that the spectral gap assumption (UG) is done for *distinct* eigenvalues: it does not preclude multiplicity. The assumption is satisfied for example for the sphere. Note that, under (UG), the geodesic flow must be periodic (see [6]), i.e., Ω is a Zoll manifold.

Remark 6 (Application of Theorem 4)**.** Theorem 4 applies in particular to the following cases:

- *The 1D torus $\mathbb{T} = \mathbb{R}/(2\pi)$.* The operator $\triangle_g = \partial_{xx}$ is defined on the subset of the functions of $H^2(\mathbb{T})$ having zero mean. All eigenvalues are of multiplicity 2 and are given by $\lambda_j = j$ for every $j \in \mathbb{N}^*$ with eigenfunctions $e_j^1(x) = \sqrt{\frac{1}{\pi}}\sin(jx)$ and $e_j^2(x) = \sqrt{\frac{1}{\pi}}\cos(jx)$. The spectral gap is $\gamma = 1$ and we compute

$$\begin{aligned}\lim_{T\to+\infty} \frac{C_T(\omega)}{T} &= \frac{1}{\pi} \inf_{j\in\mathbb{N}^*} \inf_{\alpha\in[0,1]} \int_\omega \left(\sqrt{\alpha}\sin(jx) + \sqrt{1-\alpha}\cos(jx)\right)^2 dx \\ &= \frac{1}{\pi}\left(\frac{|\omega|}{2} - \sup_{j\in\mathbb{N}^*}\sqrt{\left(\int_\omega \sin(2jx)\,dx\right)^2 + \left(\int_\omega \cos(2jx)\,dx\right)^2}\right)\end{aligned}$$

- *The unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$.* The operator $\triangle_g$ is defined from the usual Laplacian operator on the Euclidean space $\mathbb{R}^{n+1}$ by the formula $\triangle_g = r^2\triangle_{\mathbb{R}^{n+1}} - \partial_{rr} - \frac{n}{r}\partial_r$ where $r = \|x\|_{\mathbb{R}^{n+1}}$ for every $x \in \mathbb{R}^{n+1}$. Its eigenvalues are $\lambda_k = k(k+n-1)$ where $k \in \mathbb{N}$. The multiplicity of λ_k is $k(k+n-1)$ and the space of eigenfunctions is the space of homogeneous harmonic polynomials[3] of degree k. As a result, we compute

$$\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \inf_{k\in\mathbb{N}} \inf_{\phi\in\mathcal{H}_k} \frac{\int_\omega |\phi(x)|^2\,dx}{\int_{\mathbb{S}^n} |\phi(x)|^2\,dx},$$

where $\mathcal{H}_k$ is the space of homogeneous harmonic polynomials of degree k.

As a byproduct of Theorem 4, we recover a well known result on the existence of quantum limits supported by closed rays. Recall that a quantum limit for $-\triangle_g$ is a probability measure given as a weak limit (in the space of Radon measures) of the sequence of measures $(\phi_j(x)^2\,dx)_{j\in\mathbb{N}^*}$.

Corollary 5. *Under (UG), for any (closed) ray $\gamma \in \Gamma$ there exists a quantum limit supported on γ.*

This is exactly one of the main results of [20] which extends a result in [12] on the sphere. As a consequence also noted in [20], under the additional assumption that Ω is a Zoll manifold with maximally degenerate Laplacian, any measure invariant under the geodesic flow is a quantum limit. The converse is not true (see [21]).

3 Proofs

This section is devoted to prove the results stated in the latter section. In the next paragraph, we establish many results which imply all the results stated in the Introduction. More precisely,

- Theorem 1 is a consequence of Lemma 1 and Theorem 2;
- Corollary 1 is proved in Section 3.9;
- Theorem 2 is proved in Section 3.7;
- Corollary 2 is a consequence of Proposition 1;
- Corollaries 3 and 4 follow from the above the results;
- Theorem 4 is proved in Section 3.8.

[3] It is standard that an orthogonal basis of spherical harmonics can be explicitly constructed in terms of Legendre function of the first kind, the Euler's Gamma function and the hypergeometric function (see e.g. [10]).

3.1 Preliminaries and notations

Let us set $\Lambda = \sqrt{-\triangle}$. Given any $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, standing for initial conditions for the wave equation, we set

$$y^+ = \frac{1}{2}(y^0 - i\Lambda^{-1}y^1) \in L^2(\Omega) \quad \text{and} \quad y^- = \frac{1}{2}(y^0 + i\Lambda^{-1}y^1) \in L^2(\Omega). \tag{11}$$

The mapping $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \mapsto (y^+, y^-) \in L^2(\Omega) \times L^2(\Omega)$ is an isomorphism, and $\|(y^0, y^1)\|^2_{L^2\times H^{-1}} = 2(\|y^+\|^2_{L^2} + \|y^-\|^2_{L^2})$. The unique solution y of the wave equation (1) associated to the pair of initial data (y^0, y^1) belongs to $C^0(0, T; L^2(\Omega)) \cap C^1(0, T; H^{-1}(\Omega))$ and writes $y(t) = e^{it\Lambda}y^+ + e^{-it\Lambda}y^-$.

By definition, we have

$$C_T(\omega) = \frac{1}{2} \inf_{\|y^+\|^2_{L^2}+\|y^-\|^2_{L^2}=1} \int_0^T \int_\Omega \chi_\omega(x) \left|(e^{it\Lambda}y^+)(x) + (e^{-it\Lambda}y^-)(x)\right|^2 dv_g(x)\, dt.$$

Let $a : M \to \mathbb{R}$ be any measurable nonnegative function. We denote (with a slight abuse of notation) by $C_T(a)$ the quantity

$$C_T(a) = \frac{1}{2} \inf_{\|y^+\|^2_{L^2}+\|y^-\|^2_{L^2}=1} \int_0^T \int_\Omega a(x) \left|(e^{it\Lambda}y^+)(x) + (e^{-it\Lambda}y^-)(x)\right|^2 dv_g(x)\, dt.$$

This way, one has $C_T(\omega) = C_T(\chi_\omega)$.

We have

$$\begin{aligned}
&\frac{1}{T}\int_0^T \int_\Omega a|e^{it\Lambda}y^+ + e^{-it\Lambda}y^-|^2 dv_g\, dt \\
&= \frac{1}{T}\int_0^T \left(\langle ae^{it\Lambda}y^+, e^{it\Lambda}y^+\rangle + \langle ae^{-it\Lambda}y^-, e^{it\Lambda}y^-\rangle + \langle ae^{it\Lambda}y^+, e^{-it\Lambda}y^-\rangle + \langle ae^{-it\Lambda}y^-, e^{it\Lambda}y^+\rangle\right) dv_g\, dt \\
&= \left\langle \frac{1}{T}\int_0^T e^{-it\Lambda}ae^{it\Lambda}\, y^+, y^+ \right\rangle + \left\langle \frac{1}{T}\int_0^T e^{it\Lambda}ae^{-it\Lambda}\, dt\; y^-, y^- \right\rangle \\
&\qquad + \left\langle \frac{1}{T}\int_0^T e^{it\Lambda}ae^{it\Lambda}\, dt\; y^+, y^- \right\rangle + \left\langle \frac{1}{T}\int_0^T e^{-it\Lambda}ae^{-it\Lambda}\, dt\; y^-, y^+ \right\rangle
\end{aligned} \tag{12}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\Omega, v_g)$. Here, a is considered as an operator by multiplication. This formula suggests to introduce the operators $\bar{A}_T$ and $\bar{B}_T$ defined by

$$\bar{A}_T(a) = \frac{1}{T}\int_0^T e^{-it\Lambda}ae^{it\Lambda}\, dt \quad \text{and} \quad \bar{B}_T(a) = \frac{1}{T}\int_0^T e^{it\Lambda}ae^{it\Lambda} dt,$$

so that

$$C_T(a) = \inf_{\|y^+\|^2_{L^2}+\|y^-\|^2_{L^2}=1} J_T^a(y^+, y^-) \tag{13}$$

with

$$J_T^a(y^+, y^-) = \frac{1}{2}\Big(\langle \bar{A}_T(a)y^+, y^+ \rangle + \langle \bar{A}_{-T}(a)y^-, y^- \rangle + \langle \bar{B}_T(a)y^+, y^- \rangle + \langle \bar{B}_{-T}(a)y^-, y^+ \rangle \Big).$$

Given any $N \in \mathbb{N}$, we extend similarly the definition of $C_T^{>N}(\omega)$ by defining

$$C_T^{>N}(a) = \inf\{J_T^a(y^0, y^1) \mid \langle y^i, \phi_j\rangle_{(H^i)', H^i} = 0, \quad \forall i = 0, 1, \quad \forall j = 1, \dots, N \\ (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0,0)\}\}$$

and $\alpha^T(a) = \lim_{N \to +\infty} \frac{1}{T} C_T^{>N}(a)$. In what follows, the index N means that we consider initial conditions involving eigenmodes of index larger than N. More precisely, if $y \in H^{-1}(\Omega)$, $\langle y_N, \phi_j\rangle_{H^{-1}, H^1} = 0$ for every $j \leqslant N$. The same reasoning as above to obtain (13) yields

$$J_T^a(y^0, y^1) = \frac{1}{2} J_T^a(y_N^+, y_N^-). \tag{14}$$

3.2 Comments on Assumption (H)

Proposition 1. *Under* (**H**) *we have* $g_2(\mathring{\omega}) = g_2(\overline{\omega})$.

Proof. Let $\varepsilon > 0$. Without loss of generality we assume that ω is open. By definition of the infimum in the definition of $g_2^T(\omega)$, for every $\varepsilon > 0$ there exists a ray $\gamma \in \Gamma$ such that

$$\begin{aligned} g_2^T(\mathring{\omega}) + \varepsilon &\geqslant \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\, dt = \frac{1}{T}\int_0^T \chi_{\overline{\omega}}(\gamma(t))\, dt - \frac{1}{T}\int_0^T \chi_{\overline{\omega}\setminus\omega}(\gamma(t))\, dt \\ &= \frac{1}{T}\int_0^T \chi_{\overline{\omega}}(\gamma(t))\, dt + \frac{1}{T}\int_0^T \chi_{\Omega\setminus(\overline{\omega}\setminus\omega)}(\gamma(t))\, dt - 1 \\ &\geqslant g_2^T(\overline{\omega}) + g_2^T(\Omega \setminus (\overline{\omega}\setminus\omega)) - 1 \geqslant g_2^T(\overline{\omega}) \end{aligned}$$

and thus $g_2^T(\mathring{\omega}) \geqslant g_2^T(\overline{\omega})$. The converse inequality is obvious. □

3.3 Upper bound for C_T

Lemma 1. *For every Lebesgue measurable subset ω of M, one has*

$$\frac{C_T(\omega)}{T} \leqslant \min\left(\frac{1}{2} g_1(\omega), \alpha^T(\omega)\right).$$

Proof. By considering particular solutions of the form $e^{it\Lambda}\phi_j$ for a given $j \in \mathbb{N}^*$, we obtain $\frac{C_T(\omega)}{T} \leqslant \frac{1}{2} g_1(\omega)$. Besides, we have $C_T(\omega) \leqslant C_T^{>N}(\omega)$ and letting N tend to $+\infty$, we get $C_T(\omega) \leqslant \alpha^T(\omega)$. □

3.4 The high-frequency observability constant α^T

The quantity g_2^T has been defined for measurable subsets ω, but similarly to what has been done in Section 3.1, we extend its definition to arbitrary measurable nonnegative bounded functions $a : M \to \mathbb{R}$, by setting

$$g_2^T(a) = \inf_{\gamma \in \Gamma} \frac{1}{T}\int_0^T a(\gamma(t))\, dt.$$

With this notation, we have $g_2^T(\chi_\omega) = g_2^T(\omega)$, with a slight abuse of notation.

Theorem 5. *For every continuous nonnegative function $a : M \to \mathbb{R}$, we have*

$$\alpha^T(a) = \frac{1}{2} g_2^T(a).$$

Proof. We first assume that the function $a : M \to \mathbb{R}$ is smooth and thus can be considered as the symbol of an pseudo-differential $\mathrm{Op}(a)$ of order 0 corresponding to the multiplication by a. We have

$$\bar{A}_T(a) = \frac{1}{T}\int_0^T e^{-it\Lambda}\mathrm{Op}(a)e^{it\Lambda}\,dt \;\text{ and }\; \bar{B}_T(a) = \frac{1}{T}\int_0^T e^{it\Lambda}\mathrm{Op}(a)e^{it\Lambda}dt.$$

According to the Egorov theorem (see [5, 30]), the pseudo-differential operators $\bar{A}_T$ and $\bar{A}_{-T}$ are of order 0 and their principal symbols are respectively

$$\bar{a}_T = \frac{1}{T}\int_0^T a\circ\varphi_t\,dt \;\text{ and }\bar{a}_{-T} = \frac{1}{T}\int_0^T a\circ\varphi_{-t}\,dt,$$

where $(\varphi_t)_{t\in\mathbb{R}}$ is the Riemannian geodesic flow. Besides,

$$\bar{B}_T(a) = \frac{1}{T}\int_0^T e^{it\Lambda}\mathrm{Op}(a)e^{it\Lambda}\,dt \quad\text{ and }\quad \bar{B}_{-T}(a) = \frac{1}{T}\int_0^T e^{-it\Lambda}\mathrm{Op}(a)e^{-it\Lambda}\,dt$$

are pseudo-differential operators of order -1 and hence are compact (see [3, Section 3.1]).

Defining y_+ by (11) and y_N^+ as in (14), we compute (as in (12))

$$\begin{aligned}\frac{1}{T}\int_0^T\int_\Omega a|e^{it\Lambda}y_N^+ + e^{-it\Lambda}y_N^-|^2 dv_g\,dt \\ = \left\langle \bar{A}_T(a)y_N^+, y_N^+\right\rangle + \left\langle \bar{A}_{-T}(a)y_N^-, y_N^-\right\rangle + \left\langle \bar{B}_T(a)y_N^-, y_N^+\right\rangle + \left\langle \bar{B}_{-T}(a)y_N^+, y_N^-\right\rangle.\end{aligned}$$

Considering for instance the first term at the right-hand side, we have

$$\left\langle \bar{A}_T(a)y_N^+, y_N^+\right\rangle = \left\langle \frac{1}{T}\int_0^T e^{-it\Lambda}\mathrm{Op}(a)e^{it\Lambda}\,dt\; y_N^+, y_N^+\right\rangle = \langle \mathrm{Op}(\bar{a}_T)y_N^+, y_N^+\rangle + \langle K_T y_N^+, y_N^+\rangle$$

where K_T is a pseudo-differential operator of order -1 (depending on a) and thus $|\langle K_T y_N^+, y_N^+\rangle| \leqslant \|K_T\|\|y_N^+\|_{L^2}\|y_N^+\|_{H^{-1}}$. It follows from (14) that

$$\frac{1}{T}C_T^{>N}(a) = \frac{1}{2}\inf_{\|y_N^+\|_{L^2}^2+\|y_N^-\|_{L^2}^2=1}\Big(\langle \mathrm{Op}(\bar{a}_T)y_N^+, y_N^+\rangle + \langle \mathrm{Op}(\bar{a}_{-T})y_N^-, y_N^-\rangle\Big) + \mathrm{o}(1) \qquad \text{as } N\to+\infty.$$

Let us first prove that $\alpha^T(a) \geqslant \frac{1}{2}g_2^T(a)$. Denote by $S^*\Omega$ the unit cotangent bundle over Ω. By definition, we have $\bar{a}_T(x,\xi) \geqslant g_2^T(a)$ for every $(x,\xi)\in S^*\Omega$ (and similarly, $\bar{a}_{-T}(x,\xi)\geqslant g_2^T(a)$), and since the symbol $\bar{a}_T$ is real and of order 0, it follows from the Gårding inequality (see [30]) that for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\langle \mathrm{Op}(\bar{a}_T)y_N^+, y_N^+\rangle \geqslant (g_2^T(a)-\varepsilon)\|y_N^+\|_{L^2}^2 - C_\varepsilon\|y_N^+\|_{H^{-1/2}}^2$$

for every $y_N^+\in L^2(\Omega)$ (actually, one can even take $\varepsilon=0$ by using a positive quantization, for instance Op^+). Since the spectral expansion of y_N^+ involves only modes with indices larger than N, we have $\|y_N^+\|_{H^{-1/2}}^2 \leqslant \frac{1}{\lambda_N}\|y_N^+\|_{L^2}^2$ and it follows that, when considering the infimum over all possible $y_N^\pm$ of L^2 norm equal to 1, all remainder terms provide a remainder term $\mathrm{o}(1)$ as $N\to+\infty$, uniformly with respect to $y_N^\pm$. We conclude that $C_T^{>N}(a)\geqslant \frac{1}{2}g_2^T(a)+\mathrm{o}(1)$, and thus $\alpha^T(a)\geqslant\frac{1}{2}g_2^T(a)$.

Let us now prove that $\alpha^T(a)\leqslant\frac{1}{2}g_2^T(a)$. The idea is to choose some appropriate $y_N^+\in L^2(\Omega)$, and $y_N^-=0$, and to write that $\frac{1}{T}C_T^{>N}(a)\leqslant\frac{1}{2}\langle\mathrm{Op}(\bar{a}_T)y_N^+, y_N^+\rangle+\mathrm{o}(1)$. The choice of an appropriate y_N^+ is guided by the following lemma on coherent states.

Lemma 2. *Let $x_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$, and $k \in \mathbb{N}^*$. We define the coherent state*

$$u_k(x) = \left(\frac{k}{\pi}\right)^{\frac{n}{4}} e^{ik(x-x_0).\xi_0 - \frac{k}{2}\|x-x_0\|^2}.$$

Then $\|u_k\|_{L^2} = 1$, and for every symbol a on $\mathbb{R}^n$ of order 0, we have

$$\mu_k(a) = \langle \mathrm{Op}(a)u_k, u_k\rangle_{L^2} = a(x_0, \xi_0) + \mathrm{o}(1),$$

as $k \to +\infty$. In other words, $(\mu_k)_{k\in\mathbb{N}}$ converges in the sense of measures to $\delta_{(x_0,\xi_0)}$.

Admitting temporarily this (well known) lemma, we are going to define y_N^+ as an approximation of u_k, having only frequencies larger than N. Let $(x_0, \xi_0) \in S^*M$ be a minimizer of $\bar{a}_T$, i.e., $g_2^T(a) = \min \bar{a}_T = \bar{a}_T(x_0, \xi_0)$. We consider the above solution u_k, defined on M in a local chart around (x_0, ξ_0) (we multiply the above expression by a function of compact support taking the value 1 near (x_0, ξ_0), and we adapt slightly the constant so that we still have $\|u_k\|_{L^2} = 1$). Note that $\int_\Omega u_k \, dv_g = \frac{2^{\frac{n}{2}}\pi^{\frac{n}{4}}}{k^{\frac{n}{4}}}$. Now, we set

$$\pi_N u_k = \sum_{j=1}^N \langle u_k, \phi_j\rangle \phi_j = \sum_{j=1}^N \int_\Omega u_k(x)\phi_j(x)\,dx\ \phi_j dv_g(x).$$

By usual Sobolev estimates and by the Weyl law, there exists $C > 0$ such that $\|\phi_j\|_{L^\infty(\Omega)} \leqslant C\lambda_j^{\frac{n}{2}}$ and $\lambda_j \sim j^{\frac{2}{n}}$ for every $j \in \mathbb{N}^*$, hence $\|\phi_j\|_{L^\infty(\Omega)} \leqslant Cj$. We infer that

$$|\langle u_k, \phi_j\rangle| \leqslant CN \int_\Omega |u_k| \leqslant C 2^{\frac{n}{2}} \pi^{\frac{n}{4}} \frac{N}{k^{\frac{n}{4}}} dv_g(x)$$

for every $j \leqslant N$.

Let $\varepsilon > 0$ be arbitrary. Choosing k large enough so that $C 2^{\frac{n}{2}} \pi^{\frac{n}{4}} \frac{N^2}{k^{\frac{n}{4}}} \leqslant \varepsilon$, we have $\|\pi_N u_k\|_{L^2} \leqslant \varepsilon$. We set $y_N^+ = u_k - \pi_N u_k$. We have

$$\langle \mathrm{Op}(\bar{a}_T)y_N^+, y_N^+\rangle = \underbrace{\langle \mathrm{Op}(\bar{a}_T)u_k, u_k\rangle}_{\simeq g_2^T(a)} + \underbrace{\langle \mathrm{Op}(\bar{a}_T)\pi_N u_k, \pi_N u_k\rangle}_{\leqslant \varepsilon^2 \max \bar{a}_T} - \underbrace{\langle \mathrm{Op}(\bar{a}_T)\pi_N u_k, u_k\rangle}_{|\cdot| \leqslant \varepsilon \max \bar{a}_T} - \underbrace{\langle \mathrm{Op}(\bar{a}_T)u_k, \pi_N u_k\rangle}_{|\cdot|\leqslant \varepsilon \max \bar{a}_T}$$

and the conclusion follows. ∎

Proof of Lemma 2. This lemma can be found for instance in [30, Chapter 5, Example 1]. We include a proof for the sake of completeness. First of all, we compute[4] $\|u_k\|_{L^2}^2 = \left(\frac{k}{\pi}\right)^{\frac{n}{2}} \int e^{-\frac{k}{2}\|x-x_0\|^2}\,dx = 1$. Now, by definition, we have

$$\begin{aligned}\langle \mathrm{Op}(a)u_k, u_k\rangle_{L^2} = \int \mathrm{Op}(a)u_k(x)\overline{u_k(x)}\,dx &= \frac{1}{(2\pi)^n} \iiint e^{i(x-y).\xi} a(x,\xi)u_k(y)\overline{u_k(x)}\,dx\,dy\,d\xi \\ &= \frac{k^n}{(2\pi)^n} \iiint e^{ik(x-y).\xi} a(x,\xi)u_k(y)\overline{u_k(x)}\,dx\,dy\,d\xi\end{aligned}$$

[4] Here, we use the fact that $\int_{\mathbb{R}^n} e^{-\alpha\|x\|^2}\,dx = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}}$.

by the change of variable $\xi \mapsto k\xi$, and using the homogeneity of a. Then we get

$$\begin{aligned}\langle \mathrm{Op}(a)u_k, u_k\rangle_{L^2} &= \frac{k^{\frac{3n}{2}}}{2^n\pi^{\frac{3n}{2}}} \iiint a(x,\xi) e^{ik(x-y).\xi} e^{ik(y-x).\xi_0} e^{-\frac{k}{2}(\|x-x_0\|^2+\|y-x_0\|^2)}\, dx\, dy\, d\xi \\ &= \frac{k^{\frac{3n}{2}}}{2^n\pi^{\frac{3n}{2}}} \iint a(x,\xi) e^{-\frac{k}{2}\|x-x_0\|^2} \int e^{ik(x-y).\xi} e^{ik(y-x).\xi_0} e^{-\frac{k}{2}\|y-x_0\|^2}\, dy\, dx\, d\xi.\end{aligned}$$

Noting that $\mathcal{F}(e^{-\alpha\|x\|^2})(\xi) = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{\|\xi\|^2}{4\alpha}}$, we obtain

$$\begin{aligned}\int e^{ik(x-y).\xi} e^{ik(y-x).\xi_0} e^{-\frac{k}{2}\|y-x_0\|^2}\, dy &= e^{ik(x-x_0).(\xi-\xi_0)} \int e^{-ik(y-x_0).(\xi-\xi_0)} e^{-\frac{k}{2}\|y-x_0\|^2}\, dy \\ = e^{ik(x-x_0).(\xi-\xi_0)} \int e^{-iky.(\xi-\xi_0)} e^{-\frac{k}{2}\|y\|^2}\, dy &= e^{ik(x-x_0).(\xi-\xi_0)} \mathcal{F}(e^{-\frac{k}{2}\|y\|^2})(k(\xi-\xi_0)) \\ &= \left(\frac{2\pi}{k}\right)^{\frac{n}{2}} e^{ik(x-x_0).(\xi-\xi_0)} e^{-\frac{k}{2}\|\xi-\xi_0\|^2}\end{aligned}$$

and therefore,

$$\begin{aligned}\langle \mathrm{Op}(a)u_k, u_k\rangle_{L^2} &= \frac{k^n}{2^{\frac{n}{2}}\pi^n} \iint a(x,\xi) e^{ik(x-x_0).(\xi-\xi_0)} e^{-\frac{k}{2}(\|x-x_0\|^2+\|\xi-\xi_0\|^2)}\, dx\, d\xi \\ &= \frac{k^n}{2^{\frac{n}{2}}\pi^n} a(x_0,\xi_0) \iint e^{ik(x-x_0).(\xi-\xi_0)} e^{-\frac{k}{2}(\|x-x_0\|^2+\|\xi-\xi_0\|^2)}\, dx\, d\xi + \mathrm{o}(1) \\ &= c_n a(x_0,\xi_0) + \mathrm{o}(1)\end{aligned}$$

as $k \to +\infty$. Moreover, taking $a = 1$ above, we see that $c_n = \iint e^{ikx.\xi} e^{-\frac{k}{2}(\|x\|^2+\|\xi\|^2)}\, dx\, d\xi = 1$. The lemma is proved. ☐

It remains to extend the statement to the case where a is continuous only. It is obvious from the definitions of α^T and g_2^T that if $(a_k)_{k\in\mathbb{N}}$ is sequence of nonnegative smooth functions converging uniformly to a, then

$$\lim_{k\to+\infty} \alpha^T(a_k) = \alpha^T(a) \quad \text{and} \quad \lim_{k\to+\infty} g_2^T(a_k) = g_2(a).$$

Indeed, this is a consequence of the two following facts:

- the supremum of $\frac{1}{T}\int_0^T \int_\Omega |a_k - a| y^2\, dv_g\, dt$ over the set of all functions y satisfying $\|y\|_{L^2} = 1$ tends to 0 as $k \to +\infty$;
- the supremum of $\frac{1}{T}\int_0^T |a_k - a|(\gamma(t)) dt$ over the set of all rays γ tends to 0 as $k \to +\infty$.

The theorem is proved.

Remark 7. Note that $e^{it\Lambda}u_k$ (or, accordingly, $e^{it\Lambda}(u_k - \pi_N u_k)$) is a half-wave Gaussian beam along the geodesic $\varphi_t(x_0,\xi_0)$. Indeed, for any symbol of order 0, recalling that $A_t = e^{-it\Lambda}\mathrm{Op}(a)e^{it\Lambda}$ has $a_t = a \circ \varphi_t$ as principal symbol, we have $\langle \mathrm{Op}(a)e^{it\Lambda}u_k, e^{it\Lambda}u_k\rangle = \langle A_t u_k, u_k\rangle = \langle \mathrm{Op}(a_t)u_k, u_k\rangle + \mathrm{o}(1) = a_t(x_0,\xi_0) + \mathrm{o}(1)$ (by Lemma 2), which means that $e^{it\Lambda}u_k$ is microlocally concentrated around $\varphi_t(x_0,\xi_0)$.

3.5 Proof of Theorem 3

Consider an increasing sequence $(h_k)_{k\in\mathbb{N}}$ of continuous functions such that $0 \leqslant h_k \leqslant 1$ in Ω, $h_k(x) = 0$ if $\mathrm{dist}(x, \Omega \setminus \mathring{\omega}) \leqslant \frac{1}{k}$ and $h_k(x) = 1$ if $\mathrm{dist}(x, \Omega \setminus \mathring{\omega}) \geqslant \frac{2}{k}$. Note that $0 \leqslant h_k \leqslant h_{k+1} \leqslant \chi_{\mathring{\omega}}$ for every $k \in \mathbb{N}$. Let us prove that

$$g_2^T(\mathring{\omega}) = \lim_{k\to+\infty} g_2^T(h_k). \tag{15}$$

The fact that $g_2^T(\mathring{\omega}) \geqslant \limsup_{k\to+\infty} g_2^T(h_k)$ is obvious since $\chi_{\mathring{\omega}} \geqslant h_k$ for all $k \in \mathbb{N}$. Consider a sequence of rays $\gamma_k : [0,T] \to \Omega$ such that

$$g_2^T(h_k) \geqslant \frac{1}{T}\int_0^T h_k(\gamma_k(t))\,dt + \mathrm{o}(1) \quad \text{as } k \to +\infty. \tag{16}$$

The set of rays is compact since each ray is determined by it position $x \in \Omega$ at time 0 and its derivative at time 0 which lies on the unit cotangent bundle of Ω. Hence there exists $\gamma : [0,T] \to \Omega$ such that $\gamma_k \to \gamma$ uniformly on $[0,T]$. For any $t \in [0,T]$, one has

$$\liminf_{k\to+\infty} h_k(\gamma_k(t)) \geqslant \chi_{\mathring{\omega}}(\gamma(t)).$$

Indeed, if $\gamma(t) \in \mathring{\omega}$, then since $\mathring{\omega}$ is open, $h_k(\gamma_k(t)) = 1 = \chi_{\mathring{\omega}}(\gamma(t))$ as soon as k is large enough. If $\gamma(t) \notin \mathring{\omega}$, the inequality is obvious since $\chi_{\mathring{\omega}}(\gamma(t)) = 0$. By dominated convergence, we infer from (16) that

$$g_2^T(h_k) \geqslant \frac{1}{T}\int_0^T h_k(\gamma_k(t))\,dt + \mathrm{o}(1) \geqslant \frac{1}{T}\int_0 \chi_{\mathring{\omega}}(\gamma(t))\,dt + \mathrm{o}(1) \geqslant g_2^T(\mathring{\omega}) + \mathrm{o}(1) \quad \text{as } k \to +\infty,$$

which proves (15).

Using that the sequence $(h_k)_{k\in\mathbb{N}}$ is increasing and since each h_k is continuous, we obtain

$$\frac{1}{2}g_2^T(\mathring{\omega}) = \lim_{k\to+\infty} \frac{1}{2}g_2^T(h_k) = \lim_{k\to+\infty} \alpha^T(h_k) \leqslant \alpha^T(\mathring{\omega}) \leqslant \alpha^T(\omega) \leqslant \alpha^T(\bar{\omega}).$$

To conclude the proof of Theorem 3, it remains to prove that

$$\alpha^T(\bar{\omega}) \leqslant \frac{1}{2}g_2^T(\bar{\omega}). \tag{17}$$

The proof of this inequality uses exactly the same reasoning as the one used to prove $\frac{1}{2}g_2^T(\mathring{\omega}) \leqslant \alpha^T(\mathring{\omega})$. Indeed, we consider a decreasing sequence of continuous functions $(h_k)_{k\in\mathbb{N}}$ converging pointwisely to $\chi_{\bar{\omega}}$, and therefore, we have $\alpha^T(\bar{\omega}) \leqslant \alpha^T(h_k) = \frac{1}{2}g_2^T(h_k)$ and $\lim_{k\to\infty} g_2^T(h_k) = g_2^T(\bar{\omega})$. We conclude as previously that (17) is true.

3.6 Low frequencies compactness property

According to Lemma 1, one has $\frac{1}{T}C_T(\omega) \leqslant \min\left(\frac{1}{2}g_1(\omega), \alpha^T(\omega)\right)$.

Proposition 2. *If $\frac{1}{T}C_T(\omega) < \alpha^T(\omega)$ then $C_T(\omega)$ is reached, i.e., the infimum defining $C_T(\omega)$ is in fact a minimum.*

Proof. Let $(Y_k)_{k\in\mathbb{N}} = (y_k^+, y_k^-)_{k\in\mathbb{N}} \in (L^2(\Omega)\times L^2(\Omega))^{\mathbb{N}}$ be such that

$$\lim_{k\to+\infty} J_T^{\chi_\omega}(Y_k) = \frac{C_T(\omega)}{T}$$

where $J_T^{\chi_\omega}(y)$ is defined in Section 3.1 (see (13)) with $\|y_k^+\|_{L^2}^2 + \|y_k^+\|_{L^2}^2 = 1$ for every $k\in\mathbb{N}$.

Since the sequences $(y_k^\pm)_{k\in\mathbb{N}}$ are bounded in L^2, they converge weakly to an element $y_\infty^\pm \in L^2$ up to a subsequence. Therefore, we write $Y_k = Y_\infty + Z_k$ with $Y_\infty = (y_\infty^+, y_\infty^-)$ and $Z_k = (z_k^+, z_k^-)$ such that $Z_k \rightharpoonup 0$ in $L^2(\Omega)\times L^2(\Omega)$. Note that we use the norm in $L^2\times L^2$ defined by $\|(y,z)\|^2 = \|y\|_{L^2}^2 + \|z\|_{L^2}^2$. With this notations, the weak convergence of Z_k to 0 yields

$$1 = \|Y_k\|^2 = \|Y_\infty\|^2 + \|Z_k\|^2 + \mathrm{o}(1) \tag{18}$$

and

$$J_T^{\chi_\omega}(Y_k) = J_T^{\chi_\omega}(Y_\infty) + J_T^{\chi_\omega}(Z_k) + \mathrm{o}(1) \tag{19}$$

as $k\to+\infty$. To obtain (19) we have used the fact that $\langle A_T(\chi_\omega) z_k^+, y^\infty\rangle = \langle z_k^+, A_{-T}(\chi_\omega) y^\infty\rangle$ converges to to 0 by weak convergence of z_k^+ to 0 in L^2. All other crossed terms converge to 0 by using a similar argument.

Let $N\in\mathbb{N}^*$. We write $Z_k = Z_k^{\leqslant N} + Z_k^{>N}$ where $Z_k^{\leqslant N}$ is the projection on eigenmodes $j\leqslant N$. Since N is fixed, the weak convergence of Z_k to 0 implies the strong convergence of $Z_k^{\leqslant N}$ to 0. Hence, using the same reasoning as above, we obtain

$$\|Z_k\|^2 = \|Z_k^{>N}\|^2 + \mathrm{o}(1) \quad \text{and} \quad J_T^{\chi_\omega}(Z_k) = J_T^{\chi_\omega}(Z_k^{>N}) + \mathrm{o}(1)$$

as $k\to+\infty$. Using (18) and (19), we get

$$\frac{C_T(\omega)}{T} = \lim_{k\to+\infty} J_T^{\chi_\omega}(Y_k) = \lim_{k\to+\infty} \frac{J_T^{\chi_\omega}(Y_\infty) + J_T^{\chi_\omega}(Z_k^{>N}) + \mathrm{o}(1)}{\|Y_\infty\|^2 + \|Z_k^{>N}\|^2 + \mathrm{o}(1)}.$$

Assume first that $\|Y_\infty\| > 0$. Then, by definition of $C_T^{>N}(\omega)$, and $C_T(\omega)$, we obtain[5]

$$\begin{aligned}\frac{J_T^{\chi_\omega}(Y_\infty) + J_T^{\chi_\omega}(Z_k^{>N}) + \mathrm{o}(1)}{\|Y_\infty\|^2 + \|Z_k^{>N}\|^2 + \mathrm{o}(1)} &\geqslant \frac{\frac{J_T^{\chi_\omega}(Y_\infty)}{\|Y_\infty\|^2}\|Y_\infty\|^2 + \frac{C_T^{>N}(\omega)}{T}\|Z_k^{>N}\|^2 + \mathrm{o}(1)}{\|Y_\infty\|^2 + \|Z_k^{>N}\|^2 + \mathrm{o}(1)} \\ &\geqslant \min\left(\frac{J_T^{\chi_\omega}(Y_\infty)}{\|Y_\infty\|^2}, \frac{C_T^{>N}(\omega)}{T}\right) + \mathrm{o}(1).\end{aligned}$$

and therefore $\frac{C_T(\omega)}{T} \geqslant \min\left(\frac{J_T^{\chi_\omega}(Y_\infty)}{\|Y_\infty\|^2}, \frac{C_T^{>N}(\omega)}{T}\right)$. Since N is arbitrary, it follows that

$$\frac{C_T(\omega)}{T} \geqslant \min\left(\frac{J_T^{\chi_\omega}(Y_\infty)}{\|Y_\infty\|^2}, \alpha^T(\omega)\right).$$

Since $\frac{C_T(\omega)}{T} < \alpha^T(\omega)$ by assumption, we obtain

$$\frac{C_T(\omega)}{T} \geqslant \frac{J_T^{\chi_\omega}(Y_\infty)}{\|Y_\infty\|^2}$$

and therefore $\frac{C_T(\omega)}{T}$ is reached.

Assuming now that $\|Y_\infty\| = 0$, one necessarily has $\liminf_{k\to+\infty}\|Z_k\| > 0$ according to (18). The same reasoning as above yields $\frac{C_T(\omega)}{T} \geqslant \frac{C_T^{>N}(\omega)}{T}$ whenever N is large enough. It follows that $\frac{C_T(\omega)}{T} \geqslant \alpha^T(\omega)$ which is in contradiction with the assumptions. The conclusion follows. $\square$

[5] Here, we use ththe inequality $\frac{a+b}{c+d} \geqslant \min\left(\frac{a}{c}, \frac{b}{d}\right)$ for any positive real numbers a, b, c and d.

3.7 Large time asymptotics: proof of Theorem 2

According to Lemma 1, we have $\frac{1}{T}C_T(\omega) \leqslant \min\left(\frac{1}{2}g_1(\omega), \alpha^T(\omega)\right)$, and hence

$$\limsup_{T\to+\infty} \frac{C_T(\omega)}{T} \leqslant \min\left(\frac{1}{2}g_1(\omega), \alpha^\infty(\omega)\right).$$

Let us prove the converse inequality. Using the same notations as in the proof of Proposition 2, we consider a sequence $(T_k)_{k\in\mathbb{N}}$ tending to $+\infty$ and $(Y_k)_{k\in\mathbb{N}} = (y_k^+, y_k^-)_{k\in\mathbb{N}} \in (L^2(\Omega)\times L^2(\Omega))^{\mathbb{N}}$ a minimizing sequence for $\liminf_{k\to+\infty} \frac{C_{T_k}(\omega)}{T_k}$ i.e., a sequence such that

$$\lim_{k\to+\infty} J_{T_k}^{\chi_\omega}(Y_k) = \liminf_{k\to+\infty} \frac{C_{T_k}(\omega)}{T_k} \tag{20}$$

and

$$\|Y_k\|_{L^2} = 1. \tag{21}$$

We write $Y_k = Y_\infty + Z_k$ with $Y_\infty = (y_\infty^+, y_\infty^-)$ and $Z_k = (z_k^+, z_k^-)$ such that Z_k converges weakly to 0 in $L^2(\Omega)\times L^2(\Omega)$. Then

$$1 = \|Y_k\|^2 = \|Y_\infty\|^2 + \|Z_k\|^2 + \mathrm{o}(1) \tag{22}$$

and

$$J_{T_k}^\omega(Y_k) = J_{T_k}^\omega(Y_\infty) + J_{T_k}^\omega(Z_k) + \mathrm{o}(1) \tag{23}$$

as $k\to+\infty$. To obtain (23) we have used the facts that $\langle A_{T_k}(\chi_\omega)z_k^+, y^\infty\rangle = \langle z_k^+, A_{-T_k}(\chi_\omega)y^\infty\rangle$ converges to 0 by weak convergence of z_k^+ to 0 in L^2 and that $A_{-T_k}(\chi_\omega)$ converges in L^2 to $A_\infty(\chi_\omega)$ according to Lemma 4. All crossed terms converge to 0 by using a similar argument.

By Lemma 4 (see Section 3.10) and by definition of $J_{T_k}^\omega$, we get that

$$\lim_{k\to+\infty} J_{T_k}^\omega(Y_\infty) = \langle \bar{A}_\infty y_\infty^+, y_\infty^+\rangle + \langle \bar{A}_\infty y_\infty^-, y_\infty^-\rangle \geqslant g_1(\omega)\left(\|y_\infty^+\|_{L^2}^2 + \|y_\infty^-\|_{L^2}^2\right) \geqslant g_1(\omega)\|Y_\infty\|^2. \tag{24}$$

Writing $z_k = e^{it\Lambda}z_k^+ + e^{-it\Lambda}z_k^-$, we have

$$J_{T_k}^\omega(Z_k) = \frac{1}{T_k}\int_0^{T_k}\int_\omega |z_k|^2\, dv_g\, dt.$$

Let $s>0$ and write $[0,T] = [0,s]\cup[s,2s]\cup\cdots\cup[(m_k-1)s, m_k s]\cup[m_k s, T_k]$ where m_k is the integer part of T_k/s. By using several times the inequality of Footnote 5, we obtain

$$\begin{aligned}
J_{T_k}^\omega(Z_k) &= \frac{\sum_{j=0}^{m_k-1}\int_{js}^{(j+1)s}\int_\omega |z_k|^2\, dv_g\, dt + \int_{m_k s}^{T_k}\int_\omega |z_k|^2\, dv_g\, dt}{T_k} \\
&\geqslant \frac{\sum_{j=0}^{m_k-1}\int_{js}^{(j+1)s}\int_\omega |z_k|^2\, dv_g\, dt}{T_k} \\
&= \frac{\sum_{j=0}^{m_k-1}\int_{js}^{(j+1)s}\int_\omega |z_k|^2\, dv_g\, dt}{m_k s} + \left(\frac{1}{T_k} - \frac{1}{m_k s}\right)\int_0^{m_k s}\int_\omega |z_k|^2\, dv_g\, dt \\
&\geqslant \min_{1\leqslant j\leqslant m_k} \frac{\int_{js}^{(j+1)s}\int_\omega |z_k|^2\, dv_g\, dt}{s} + \left(\frac{1}{T_k} - \frac{1}{m_k s}\right)\int_0^{m_k s}\int_\omega |z_k|^2\, dv_g\, dt.
\end{aligned}$$

Using that $0 \leqslant m_k s - T_k < s$, that $T_k \to +\infty$ and that

$$\int_0^{m_k s} \int_\omega |z_k|^2 \, dv_g \, dt \leqslant \int_0^{m_k s} \int_\Omega |z_k|^2 \, dv_g \, dt = m_k s \|Z_k\|^2 \leqslant (1 + \|Y_\infty\|^2) m_k s,$$

we get

$$J_{T_k}^\omega(Z_k) \geqslant \min_{1 \leqslant j \leqslant m_k} J_{\omega,s}(\tilde{z}_{k,j}^+, \tilde{z}_{k,j}^-) + \mathrm{o}(1) \quad \text{with} \quad J_{\omega,s}(\tilde{z}_{k,j}^+, \tilde{z}_{k,j}^-) = \frac{1}{s} \int_0^s \int_\omega |\tilde{z}_{k,j}|^2 \, dv_g \, dt$$

where $(\tilde{z}_{k,j}^+, \tilde{z}_{k,j}^-)$ is the initial condition associated to the solution $z_{k,j} : (t, x) \mapsto z_k(t + js, x)$.

Proceeding as in the proof of Proposition 2 and decomposing $\tilde{Z}_{k,j} = (\tilde{z}_{k,j}^+, \tilde{z}_{k,j}^-)$ in low/high frequencies as before, we get that, for any nonzero integer N,

$$J_{\omega,s}(\tilde{z}_{k,j}^+, \tilde{z}_{k,j}^-) \geqslant \frac{C_s^N(\omega)}{s} \|\tilde{Z}_{k,j}\|^2 + \mathrm{o}(1).$$

Since the wave group is unitary, one has $\|\tilde{Z}_{k,j}\|^2 = \|Z_k\|^2$ and hence

$$J_{\omega,s}(\tilde{z}_{k,j}^+, \tilde{z}_{k,j}^-) \geqslant \frac{C_s^N(\omega)}{s} \|Z_k\|^2 + \mathrm{o}(1).$$

Combining these last facts with (21), (22), (23) and (24), we obtain

$$\liminf_{k \to +\infty} \frac{C_{T_k}(\omega)}{T_k} \geqslant \frac{g_1(\omega) \|Y_\infty\|^2 + \frac{C_s^N(\omega)}{s} \|Z_k\|^2 + \mathrm{o}(1)}{\|Y_\infty\|^2 + \|Z_k\|^2 + \mathrm{o}(1)} \geqslant \min\left(g_1(\omega), \frac{C_s^N(\omega)}{s}\right) + \mathrm{o}(1).$$

Since N is arbitrary, we obtain $\liminf_{k \to +\infty} \frac{C_{T_k}(\omega)}{T_k} \geqslant \min(g_1(\omega), \alpha^s(\omega))$, and since s is arbitrary, we conclude that

$$\liminf_{k \to +\infty} \frac{C_{T_k}(\omega)}{T_k} \geqslant \min(g_1(\omega), \alpha^\infty(\omega)).$$

It remains to show the last claim of the theorem. Let us assume that $g_1(\omega) < \alpha^\infty(\omega)$. Let us assume by contradiction that $\frac{1}{2} g_1(\omega)$ is not reached. Then, there exists a subsequence $(\phi_{j_k})_{k \in \mathbb{N}}$ of eigenfunctions of $-\triangle_g$ normalized in $L^2(\Omega)$ such that $j_k \to +\infty$ and $\frac{1}{2} g_1(\omega) = \int_\omega \phi_{j_k}(x)^2 \, dv_g + \mathrm{o}(1)$ as $k \to +\infty$. Now, by definition of $\alpha^\infty(\omega)$, by taking $Y_k = (\phi_{j_k}, 0)$ as initial condition in the infimum defining $C_T^{>N}(\omega)$, we infer that $\frac{1}{T} C_T^{>N}(\omega) \leqslant J_T^\omega(Y_k)$ provided that k be large enough. Passing to the limit with respect to N and T yields $\alpha^\infty(\omega) \leqslant \frac{1}{2} g^1(\omega)$, which is a contradiction.

3.8 Large time asymptotics under the condition (UG): proof of Theorem 4

The proof follows the same lines as the one of Theorem 3.7. Using the same notations, we have

$$\lim_{k \to +\infty} J_{T_k}^{\chi_\omega}(Y_k) = \liminf_{k \to +\infty} \frac{C_{T_k}(\omega)}{T_k}$$

and

$$1 = \|Y_k\|^2 = \|Y_\infty\|^2 + \|Z_k\|^2 + \mathrm{o}(1), \qquad J_{T_k}^\omega(Y_k) = J_{T_k}^\omega(Y_\infty) + J_{T_k}^\omega(Z_k) + \mathrm{o}(1),$$

and moreover,

$$\lim_{k \to +\infty} J_{T_k}^\omega(Y_\infty) = \langle \bar{A}_\infty y_\infty^+, y_\infty^+ \rangle + \langle \bar{A}_\infty y_\infty^-, y_\infty^- \rangle \geqslant g_1(\omega) \left(\|y_\infty^+\|_{L^2}^2 + \|y_\infty^-\|_{L^2}^2 \right) \geqslant g_1(\omega) \|Y_\infty\|^2.$$

Using Lemma 5 (see Section 3.10), we infer that

$$\lim_{k\to+\infty} J^{\omega}_{T_k}(Z_k) = \langle \bar{A}_\infty z_k^+, z_k^+\rangle + \langle \bar{A}_\infty z_k^-, z_k^-\rangle \geqslant g_1(\omega)\left(\|z_k^+\|^2_{L^2} + \|z_k^-\|^2_{L^2}\right) \geqslant g_1(\omega)\|Z_k\|^2,$$

and thus

$$\liminf_{k\to+\infty} \frac{C_{T_k}(\omega)}{T_k} \geqslant \frac{g_1(\omega)\|Y_\infty\|^2 + g_1(\omega)\|Z_k\|^2 + \mathrm{o}(1)}{\|Y_\infty\|^2 + \|Z_k\|^2 + \mathrm{o}(1)}.$$

The conclusion follows.

3.9 Characterization of observability: proof of Corollary 1

We first observe that $C_T(\omega) > 0$ implies that $\alpha^T(\omega) > 0$. Indeed, since $C_T(\omega) \leqslant C_T^{>N}(\omega)$ for every $N \in \mathbb{N}^*$, it follows from the definition of α^T that $\alpha^T(\omega) = 0 \Rightarrow C_T(\omega) = 0$.

Let us prove the converse. Assume by contradiction that

$$\alpha^T(\omega) > 0 \qquad \text{and} \qquad C_T(\omega) = 0. \tag{25}$$

For any $s > 0$, let us denote by E_s the vector space (sometimes called "space of invisible solutions") of initial data $Y = (y^+, y^-)$ in $L^2(\Omega) \times L^2(\Omega)$ such that $e^{it\Lambda}y^+ e^{-it\Lambda}y^-$ vanishes identically on $[0, s] \times \omega$.

We claim that the following property holds true for every $k \in \mathbb{N}$:

(H_k) For every $\varepsilon > 0$ there exists a non trivial $Y_{k,\varepsilon} = (y^+_{k\varepsilon}, y^-_{k\varepsilon}) \in E_{T-\varepsilon}$ involving only frequencies of index greater than k, i.e., such that

$$\int_\Omega y^{\pm}_{k,\varepsilon}(x)\phi_j(x)\, dv_g(x) = 0, \qquad i = 0, 1, \quad j = 1, \dots, k.$$

If $k = 0$ this property writes: there exists a non trivial solution $Y_{0,\varepsilon} \in E_{T-\varepsilon}$.

Admitting this fact temporarily, if $\varepsilon > 0$ and N are fixed, Property (H_N) yields the existence of $Y_{T,\varepsilon} = (y^+_{T\varepsilon}, y^-_{T\varepsilon}) \in E_{T-\varepsilon}$ involving only frequencies of index higher than N such that $e^{it\Lambda}y^+_{T,\varepsilon} + e^{-it\Lambda}y^-_{T,\varepsilon}$ vanishes identically on $[0, T-\varepsilon] \times \omega$. Using $Y_{T,\varepsilon}$ as test functions in the functional $J^{\chi_\omega}_T$, one infers that $C^{>N}_{T-\varepsilon}(\omega) = 0$. Note that, without loss of generality, we may assume that $\|Y_{T,\varepsilon}\| = 1$. Letting N tend to $+\infty$ yields that $\alpha^{T-\varepsilon}(\omega) = 0$. Finally, noting that for all (y^+, y^-) of norm 1, one has

$$\left|J^{\chi_\omega}_{T-\varepsilon}(y^+, y^-) - J^{\chi_\omega}_T(y^+, y^-)\right| \leqslant \frac{\varepsilon}{T-\varepsilon},$$

we infer that $\alpha^T(\omega) \leqslant \alpha^{T-\varepsilon}(\omega) + \frac{\varepsilon}{T-\varepsilon}$ and thus $\alpha^T(\omega) = 0$, whence the contradiction.

Let us now prove by recurrence that Property (H_k) holds true for every $k \in \mathbb{N}$ under the assumption (25). Let us first prove that (H_0) is true. According to Theorem 1, the infimum defining $C_T(\omega)$ in Definition (13) is reached by some $Y = (y^+, y^-)$ such that $e^{it\Lambda}y^+_{T,\varepsilon} + e^{-it\Lambda}y^-_{T,\varepsilon}$ vanishes identically on $[0, T] \times \omega$. In other words, the dimension of E_T is at least equal to 1, and this is also true for $E_{T-\varepsilon}$ for any ε since $E_T \subset E_{T-\varepsilon}$.

Assume now that (H_k) is true for some $k \in \mathbb{N}$ and let us show that (H_{k+1}) is also true. Let $\varepsilon > 0$ and let $Y = (y^+, y^-) \in E_{T-\varepsilon/2}$ satisfying

$$\int_\Omega y^{\pm}(x)\phi_j(x)\, dv_g(x) = 0, \qquad \text{for all } i = 0, 1, \quad j = 1, \dots, k.$$

Define $y(t,\cdot) = e^{it\Lambda}y^+ + e^{-it\Lambda}y^-$. The crucial point is that for every $s \in [0,\varepsilon/2]$, the function $\tau_s(y) : (t,x) \to y(t+s,x)$ belongs to $E_{T-\frac{\varepsilon}{2}-s}$ which is contained in $E_{T-\varepsilon}$.

We now show the existence a $Z = (z^+, z^-)$ such that the function

$$z : (t,x) \mapsto e^{it\Lambda}z^+(x) + e^{-it\Lambda}z^-(x) \tag{26}$$

which is a nonzero linear combination of functions $(\tau_s(y))_{s\in[0,\varepsilon/2]}$, satisfies the orthogonality condition

$$\int_\Omega z^\pm(x)\phi_j(x)\,dv_g(x) = 0, \qquad i = 0,1, \quad j = 1,\dots k+1.$$

We expand the solution $\tau_s(y)$ as

$$\tau_s(y)(t,\cdot) = \sum_{j=k+1}^{+\infty} \left(a_j(s)e^{i\lambda_j t} + b_j(s)e^{-i\lambda_j t}\right)\phi_j(\cdot)$$

where $(a_j(s))_{j\in\mathbb{N}^*}$ and $(b_j(s))_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{R})$. In particular, we have

$$a_j(s) = e^{is\lambda_j}a_j(0) \text{ and } b_j(s) = e^{-is\lambda_j}b_j(0).$$

If $a_{k+1}(0) = b_{k+1}(0) = 0$ then y belongs to $E_{T-\varepsilon}$ and involves only frequencies of index higher than $k+1$ which shows that (H_{k+1}) holds true. For this reason, we assume that $a_{k+1}(s) \neq 0$ or $b_{k+1}(s) \neq 0$. Hence, there exists j such that $\lambda_j > \lambda_{k+1}$, and $a_j(0) \neq 0$ or $b_j(0) \neq 0$. Otherwise, the function y would be a nonzero multiple of an eigenfunction belonging to the eigenspace associated to the eigenvalue λ_k and would vanish on ω: but this is impossible as soon as ω has a positive Lebesgue measure (see [4, 8, 19]), which is the case since $\alpha^T(\omega) > 0$. Hence, let us consider $j > k$ such that $\lambda_j > \lambda_k$ and $a_j(0) \neq 0$ or $b_j(0) \neq 0$. Since $\lambda_j > \lambda_k$, one can find $0 < s < s' \leqslant \varepsilon/2$ such that the vectors $(1, e^{i\lambda_k s}, e^{i\lambda_k s'})$ and $(1, e^{i\lambda_j s}, e^{i\lambda_j s'})$ are linearly independent. In other words, there exist real numbers $c_0, c_s, c_{s'}$ such that

$$c_0 + c_s e^{i\lambda_k s} + c_{s'}e^{i\lambda_k s'} = 0 \tag{27}$$

and

$$c_0 + c_s e^{i\lambda_j s} + c_{s'}e^{i\lambda_j s'} \neq 0. \tag{28}$$

Then $z = c_0 y + c_s y_s + c_{s'}y_{s'}$ is the desired solution. Indeed, writing it as in (26), we obtain $Z \in E_{T-\varepsilon}$ and moreover $z \neq 0$ by (28). Finally, z involves only frequencies of index larger than $k+1$ by (27). This shows (H_{k+1}).

3.10 Convergence properties for $\bar{A}_T$ and $\bar{B}_T$

In this section, we establish some convergence properties as $T \to \infty$ for the operators $\bar{A}_T(a)$ and $\bar{B}_T(a)$ introduced in Section 3.1. We recall that $(\lambda_j)_{j\geqslant 1}$ denotes the sequence of eigenvalues of $\Lambda = \sqrt{-\triangle_g}$ counted with multiplicity and that $(\phi_j)_{j\geqslant 1}$ is an orthonormal L^2-basis of eigenfunctions of $-\triangle_g$ such that ϕ_j is associated to λ_j^2. Now, let P_j be the L^2-projector defined by $P_j y = \langle y, \phi_j\rangle\phi_j$.

Throughout this section, let a be a bounded nonnegative measurable function, considered as an operator by multiplication.

Lemma 3. *We have*

$$\bar{A}_T(a) = \sum_{j,l\geqslant 0} f_T(\lambda_j - \lambda_l) P_j a P_l \quad \textit{and} \quad \bar{B}_T(a) = \sum_{j,l\geqslant 0} f_T(\lambda_j - \lambda_l) P_j a P_l$$

where $f_T(x) = \begin{cases} \frac{e^{iTx}-1}{iTx} & \textit{if } x \neq 0; \\ 1 & \textit{if } x = 0. \end{cases}$.

Proof. Let $y \in L^2(\Omega)$. We set $y_j = \langle y, \phi_j \rangle$ so that $y = \sum_j y_j \phi_j$. We have

$$\bar{A}_T(\omega)y = \sum_j \langle \bar{A}_T(a)y, \phi_j \rangle \phi_j = \sum_j \left(\sum_l \frac{1}{T} \int_0^T e^{it(\lambda_j - \lambda_l)} \, dt y_j \int_\Omega a\phi_j \phi_l \, dv_g \right) \phi_l$$

and $\frac{1}{T}\int_0^T e^{it(\lambda_j-\lambda_l)}\, dt = f_T(\lambda_j - \lambda_l)$. A similar reasoning is done for $\bar{B}_T(a)$. □

Lemma 4. *For every* $y = \sum_j P_j y = \sum_j y_j \phi_j \in L^2(\Omega)$*, we have*

$$\bar{A}_T(a)y = \frac{1}{T}\int_0^T e^{-it\Lambda} a e^{it\Lambda} \, dt \ y \underset{T\to\pm\infty}{\longrightarrow} \bar{A}_\infty(a)y = \sum_j \left(y_j \int_\Omega a\phi_j^2 \, dv_g \right) \phi_j$$

and

$$\bar{B}_T(a)y = \frac{1}{T}\int_0^T e^{-it\Lambda} a e^{it\Lambda} \, dt \ y \underset{T\to\pm\infty}{\longrightarrow} 0.$$

In other words, the operator $\bar{A}_T(a)$ *(resp.* $\bar{B}_T(a)$*) converges pointwisely to a diagonal operator (resp. 0) in* $L^2(\Omega)$ *as* $T \to \pm\infty$.

Proof. Let l be a fixed integer. We first show that

$$\lim_{T\to\pm\infty} \langle A_T(a)y, \phi_l \rangle = \langle A_\infty(a)y, \phi_l \rangle \tag{29}$$

Let $N \in \mathbb{N}$. Setting

$$r_N = \sum_{j>N} \frac{y_j}{T} \int_0^T e^{it(\lambda_j - \lambda_l)} \, dt \int_\Omega a\phi_j \phi_l dv_g \ \in \mathbb{C},$$

we have

$$\langle \bar{A}_T(a)y, \phi_l \rangle = \sum_{j\leqslant N} f_T(\lambda_j - \lambda_l) y_j \int_\Omega a(x)\phi_j\phi_l dv_g(x) + r_N.$$

If $\lambda_j \neq \lambda_l$ then $f_T(\lambda_j - \lambda_l) \to 0$ as $T \to \pm\infty$, and if $\lambda_j = \lambda_l$ then $f_T(\lambda_j - \lambda_l) = 1$. Therefore the limit of the finite sum above is equal to $y_l \int_\Omega a(x)\phi_l^2 dv_g(x)$. Let us prove that r_N is arbitrarily small if N is large enough.

Setting $y^N = \sum_{j>N} y_j \phi_j$ (high-frequency truncature) and considering $C > 0$ such that $a \leqslant C$ a.e. in Ω, we have

$$|r_N| = \left| \frac{1}{T}\int_0^T \int_\Omega \sum_{j>N} e^{it\lambda_j} y_j \phi_j(x) e^{-it\lambda_l} \phi_l(x) \, dv_g(x) \, dt \right| = \left| \frac{1}{T}\int_0^T \int_\Omega a(x)(e^{it\Lambda}y^N)(x) e^{-itl}\phi_l(x) \, dv_g(x) \, dt \right|$$

$$\leqslant \frac{C}{T}\int_0^T \int_\Omega |(e^{it\Lambda}y^N)(x)||\phi_l(x)| \, dv_g(x) \, dt \leqslant \left(\frac{1}{T}\int_0^T \|e^{it\Lambda}y^N\|_{L^2}^2 \, dt \right)^{1/2} = \|y^N\|_{L^2}^2$$

since $e^{it\Lambda}$ is an isometry in $L^2(\Omega)$. Therefore $r_N = \mathrm{o}(1)$ as $N \to +\infty$.

We have proved that $\langle \bar{A}_T(a)y, \phi_l\rangle \to y_l \int_\Omega \phi_l^2 dv_g(x)$ as $T \to \pm\infty$ and then (29) is true. It follows that $\bar{A}_T(a)y \rightharpoonup \bar{A}_\infty(a)y$ for the weak topology of $L^2(\Omega)$.

Let us now write $y = y_N + y^N$ with $y_N = \sum_{j\leqslant N} y_j\phi_j$ and $y^N = \sum_{j>N} y_j\phi_j$. By compactness for frequencies lower than or equal to N, we have $\bar{A}_T(a)y_N \to \bar{A}_\infty(a)y_N$ for the strong topology of $L^2(\Omega)$. Besides, noting that $\|\bar{A}_T(a)\| \leqslant 1$, we have $\|\bar{A}_T(a)y^N\| \leqslant \|y^N\|$, and since $\|y^N\|$ can be made arbitrarily small by taking N large, the result follows.

The same argument allows to prove that $\bar{B}_T(a)y$ tends to 0 when $T \to \pm\infty$. □

Lemma 5. *Under (UG), $\bar{A}_T(a)$ converges uniformly (i.e., in operator norm) to $\bar{A}_\infty(a)$ as $T \to \pm\infty$.*

Proof. It suffices to prove that

$$\lim_{T\to+\infty} \sup_{\sum_j |y_j|^2=\sum_l |z_l|^2=1} \sum_{j\neq l} f_T(\lambda_j-\lambda_l)\langle a\phi_j,\phi_l\rangle y_l z_l = 0.$$

Since $|f_T(\lambda_j-\lambda_l)| \leqslant \frac{2}{T|\lambda_j-\lambda_l|}$, we have

$$\left|\sum_{j\neq l} f_T(\lambda_j-\lambda_j)\langle a\phi_j,\phi_l\rangle y_j z_l\right| | \leqslant \frac{2}{T}\sum_{j\neq l}\frac{|y_j||z_l|}{|\lambda_j-\lambda_l|} \leqslant \frac{\mathrm{C}}{T},$$

as a consequence of Montgomery-Vaughan's inequality (recalled below) and where $C>0$ is independent of $(y_j)_{j\in\mathbb{N}}$, $(z_l)_{l\in\mathbb{N}}$, $(\phi_j)_{j\in\mathbb{N}}$, $(\phi_l)_{l\in\mathbb{N}}$. The result follows. □

The well known Hilbert inequality states that

$$\left|\sum_{j\neq k}\frac{a_j\bar{b}_k}{j-k}\right|^2 \leqslant \pi^2\sum_{j=1}^{+\infty}|a_j|^2\sum_{j=1}^{+\infty}|b_j|^2 \qquad \forall (a_j)_{j\in\mathbb{N}},(b_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{C}).$$

The same statement holds true with $j-k$ replaced with $j+k$. A generalization by Montgomery and Vaughan in [23] states that, given $\lambda_1 < \cdots < \lambda_j < \cdots$ with $\lambda_{j+1}-\lambda_j \geqslant \delta > 0$ for every j (uniform gap), one has

$$\left|\sum_{j\neq k}\frac{a_j\bar{b}_k}{\lambda_j-\lambda_k}\right|^2 \leqslant \frac{\pi^2}{\delta^2}\sum_{j=1}^{+\infty}|a_j|^2\sum_{j=1}^{+\infty}|b_j|^2 \qquad \forall (a_j)_{j\in\mathbb{N}},(b_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{C}).$$

4 Concluding remarks and perspectives

We provide here a list of open problems and issues.

Manifolds with boundary. The introduction of the so-called *high-frequency observability constant* $\alpha^T(\omega)$ is of interest because of the equivalence $C_T(\omega)>0 \Leftrightarrow \alpha^T(\omega)>0$ stated in Corollary 1. It is still true on a manifold with boundary. But then extending Theorem 3 and Corollary 3 to manifolds with boundary raises difficulties.

Schrödinger equation. It is known that GCC implies internal observability of the Schrödinger equation (see [16]), but this sufficient condition is not sharp (see [11]). Until now a necessary and sufficient condition for observability is still not known (see [13]). We think that some of the approaches developed in this paper, combined with microlocal issues, may serve to address this problem.

Shape optimization. A challenging problem is to maximize the functional $\omega \mapsto C_T(\omega)$ over the set of all possible measurable subsets of Ω of measure $|\omega| = L|\Omega|$ for some fixed $L \in (0,1)$. In [24, 26], the maximization of the *randomized* observability constant has been considered, that is, the functional $\omega \mapsto g_1(om)$. Maximizing the functional $\omega \mapsto g_2(\omega)$ is an interesting open problem which, thanks to Corollary 4, would be a step towards the maximization of the *deterministic* observability constant.

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